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An update on Minc's survey of open problems involving permanents ☆

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Abstract

We summarise the progress which has been made since 1986 on the conjectures and open problems listed in H. Minc's survey articles on the theory of permanents.

Keywords: Permanent; Subpermanent sum; Doubly stochastic; Positive semidefinite Hermitian; Permanental dominance conjecture; Circulant

1. Introduction

Henryk Minc published a series of extremely useful survey articles [58,69,70] and one excellent book [68] in which he summarised the state-of-the-art in the theory of permanents at the time of writing, concentrating particularly on progress since his last report. A noteworthy feature of these surveys was a catalogue of conjectures and open problems which spurred many advances in the field. It is now two decades

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since Minc last updated his catalogue, and many discoveries have been made over the intervening years. Hence the present authors thought it an appropriate time to write another progress report.

While we would like to stay true to the spirit of Minc's work, it is unfortunately no longer practical to attempt to list all the published papers on permanents. Any such hope is defeated by the sheer volume of such works, which since 1986 we estimate to be well in excess of a thousand papers. Thus we have chosen to concentrate on the conjectures and unsolved problems, and to survey only those papers which are of direct relevance to their solution.

2. Notation and terminology

Throughout this work the following notation and terminology will be used. S_n will denote the symmetric group on $\{1, 2, \ldots, n\}$ and $\mathbf{1}_n$ will denote the identity permutation in S_n. The *permanent* of an $n \times n$ matrix $A = [a_{ij}]$ is defined by

$$\mathrm{per}(A) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

The kth *subpermanent sum*, $\sigma_k(A)$, is defined to be the sum of the permanents of all order k submatrices of A.

The direct sum of t copies of A will be denoted $\sum^t A$. The direct sum of A and B will be denoted $A \oplus B$, while their tensor product will be denoted $A \otimes B$ and their Hadamard (elementwise) product will be denoted $A * B$. We say that A and B are *permutation equivalent* if there exist permutation matrices P and Q such that $A = PBQ$.

The following notation will be used for sets of $n \times n$ matrices: $\mathscr{H}_n$ will denote the set of positive semi-definite Hermitian matrices and Ω_n will denote the set of doubly stochastic matrices. The set of non-negative matrices with each row and column sum equal to k will be denoted $\overline{\Lambda}_n^k$. The subset of all (0, 1)-matrices in $\overline{\Lambda}_n^k$ will be denoted Λ_n^k. The subset consisting of all circulant matrices in Λ_n^k will be denoted Δ_n^k.

For any matrix A the Euclidean norm of A will be denoted $\|A\|$ and the submatrix obtained by deleting row i and column j from A will be denoted $A(i|j)$. The Hermitian adjoint (conjugate transpose) of A will be denoted A^*. The conjugate of a complex number c will also be denoted c^*.

Let $Z = [z_{ij}]$ be a (0, 1)-matrix of order n. The *complementary matrix* $\overline{Z} = [\overline{z}_{ij}]$ is defined by $\overline{z}_{ij} = 1 - z_{ij}$ for $1 \leqslant i, j \leqslant n$. The face of the doubly stochastic polytope defined by Z, denoted by $\Omega(Z)$, is

$$\Omega(Z) = \left\{A = [a_{ij}] \in \Omega_n : a_{ij} \leqslant z_{ij} \text{ for all } i, j\right\}.$$

I_n is the identity and $D_n = \overline{I}_n$ is its complement. P_n is the permutation matrix corresponding to $(1234 \cdots n)$. $\mathscr{J}_n = [\frac{1}{n}]$ is the matrix in Ω_n in which all entries are equal, while $J_n = n\mathscr{J}_n$ denotes the all 1 matrix. More generally, $J_{s,t}$ denotes an $s \times t$ block of ones.

3. The permanental dominance conjecture

The permanental dominance conjecture (Conjecture 42) has, arguably, adopted the mantle from the van der Waerden conjecture as the most actively pursued prize among H. Minc's catalogue of unsolved problems. In this section we review some of the highlights of progress on Conjecture 42 since 1986. Any reader who is interested in further details is encouraged to seek out the expository articles by Merris [66,67], James [41,42] and Pate [78,80,81] among others.

In a historical parallel with the van der Waerden conjecture, attempts to solve the permanental dominance conjecture have spawned a number of interesting conjectures. Some of these have become important goals in their own right and a number are included within Minc's catalogue. The known relationships between Conjectures 9, 30, 31, 32, 38, 40 and 42 are shown by the implications in Fig. 1. For proofs of these relationships, the reader should consult [1,2,66,78]. It is also worth noting that Problem 2 in Minc's catalogue asks for the resolution of a specific case of the permanental dominance conjecture.

If G is a subgroup of S_n and χ is any character of G then the *generalized matrix function* f_χ is defined by

$$f_\chi(M) = \sum_{\sigma\in G} \chi(\sigma) \prod_{i=1}^{n} m_{i,\sigma(i)},$$

for each $n \times n$ complex matrix $M = [m_{ij}]$. If $M \in \mathscr{H}_n$ then f_χ is a non-negative real number. If $G = S_n$ and χ is irreducible then f_χ is called an *immanant*. If χ is the principal/trivial character then f_χ is the permanent, while if χ is the alternating character then f_χ is the determinant.

The permanental dominance conjecture asserts that $\mathrm{per}(A) \geqslant f_\chi(A)/\chi(\mathbf{1}_n)$ for all $A \in \mathscr{H}_n$, irrespective of the choice of χ. It is the permanental analogue of a classical result of Schur [85] which says that $\det(A) \leqslant f_\chi(A)/\chi(\mathbf{1}_n)$ for all $A \in \mathscr{H}_n$.

$$\begin{array}{ccccccc}
 & & 42 & \Longrightarrow & 9 & & \\
 & & \Uparrow & & & & \\
32 & \Longleftrightarrow & 31 & \Longrightarrow & 38 & \Longrightarrow & 30 \\
 & & \Downarrow & & & & \\
 & & 40 & & & &
\end{array}$$

Fig. 1. Relationship between Conjectures 9, 30, 31, 32, 38, 40 and 42.

Interestingly, James [42] discovered that the following matrix in H_4:

$$\begin{pmatrix} \sqrt{3} & i & i & -i \\ -i & \sqrt{3} & i & i \\ -i & -i & \sqrt{3} & -i \\ i & -i & i & \sqrt{3} \end{pmatrix}$$

achieves equality in the permanental dominance inequality in the case where G is the alternating group A_4 and χ is one of the characters of that group.

There has been little recent progress made on the permanental dominance conjecture in its full generality, with most authors content to attack its specialisation to immanants. Here significant progress has been made, although the main result has so far proved elusive.

To describe the results which have been proved we will find it convenient to follow Pate [75] in defining a partial order $\preceq$ on the set of partitions of an integer n. Let λ and μ be two such partitions and let χ and χ' be the characters associated with λ and μ respectively by the well known bijection between partitions of n and irreducible characters of S_n. By $\lambda \preceq \mu$ we will mean that for all $H \in \mathscr{H}_n$,

$$\frac{f_\chi(H)}{\chi(\mathbf{1}_n)} \leqslant \frac{f_{\chi'}(H)}{\chi'(\mathbf{1}_n)}.$$

For two partitions λ, μ of n to satisfy $\lambda \preceq \mu$ it is necessary but not sufficient that μ majorizes λ.

The result of Schur quoted above implies that $(1^n) \preceq \lambda$ for all partitions λ of n. The specialisation of the permanental dominance conjecture to immanants asserts that for all λ,

$$\lambda \preceq (n). \tag{1}$$

A significant step was made by Heyfron [35], who neatly resolved the case of the so-called "single-hook immanants" by showing that

$$(1^n) \preceq (2, 1^{n-2}) \preceq (3, 1^{n-3}) \preceq \cdots \preceq (n-1, 1) \preceq (n). \tag{2}$$

This confirmed a conjecture originally made by Merris [65]. Numerous special cases of the inequalities implied by (2) had been shown by Johnson and Pierce [45,46] prior to Heyfron's proof.

James and Liebeck [43] showed that (1) holds whenever λ has at most two parts which exceed 1. The slightly weaker result that (1) holds whenever λ has exactly two parts was subsequently obtained by Pate [73] who, notably, obtained his result by proving a special case of Soules' conjecture (Conjecture 31). Pate then improved his result successively to show that (1) holds when (i) λ has at most two parts which exceed two [77], (ii) λ has at most three parts which exceed two [80], (iii) λ has at most four parts which exceed two [81], provided that the second and third parts are equal in the case when there are four.

As a corollary of this last result, it follows that (1) is true whenever $n \leqslant 13$. This improved on the $n \leqslant 7$ observed earlier by James [42] and the $n \leqslant 9$ observed by Pate [77]. A general scheme for obtaining inequalities involving immanants is described by Pate in [79].

Pate [81] showed the following results for positive integers n, p and k. If $k \geqslant 2$ and $n \geqslant p+k-2$ then $(n+p-i, n^k, i) \preceq (n+p, n^k)$ for $1 \leqslant i \leqslant p$. On the other hand if $p \geqslant n+k-1$ then $(n+p-i, n^k, i) \preceq (n+p, n^k)$ whenever $p/2 \leqslant i \leqslant n$. In the same paper he obtained the following asymptotic result. For positive integers k and s there exists an integer $N_{k,s}$ such that for all $n \geqslant N_{k,s}$,

$$(n+s, n^k) \preceq (2n+s, n^{k-1}) \preceq (3n+s, n^{k-2}) \preceq \cdots \preceq (kn+n+s).$$

A common way to represent a partition $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ is by means of its *Young diagram* (also known as a *Ferrers Diagram*), which consists of left-justified rows of boxes, with α_i boxes in row i. An approach which has proved conceptually useful is to consider the effect that various operations on the Young diagram have on the ranking of a partition in the partial order ($\preceq$). Pate has shown that each of the following operations produce a partition which is lower in the partial order:

1. Moving all the boxes in the last column into the first column [75].
2. Moving all the boxes in any column other than the first into the first column [74].
3. Moving a single corner box into the first column [76]. By a corner box we mean a box which has no box directly below or to the right of it in the diagram. For example, Fig. 2 shows the Young diagram for the partition $(6, 3^2, 1)$, with the three corner boxes marked with an X. By moving the corner box from the third column into the first column we create the partition $(6, 3, 2, 1^2)$ (Fig. 3), and Pate's result implies that $(6, 3, 2, 1^2) \preceq (6, 3^2, 1)$.

Note that each of the operations 1, 2 and 3 is more general than its predecessors (repeated application of 3 can produce the same effect as 2), but that even operation 1 is powerful enough to imply (2) and the result that all immanants dominate the determinant.

Another natural operation on partitions is to combine two parts (then reorder the parts if necessary, to achieve a non-increasing sequence). Operation 3 above shows that combining a part of size 1 with any other part increases the ranking of a partition in the $\preceq$ ordering. Pate [77] showed that the same effect is achieved by combining a part of size 2 with the largest part. It seems quite plausible that combining any two parts in a partition will increase the ranking of the partition, and that proving this might be the easiest route to proving the permanental dominance conjecture for

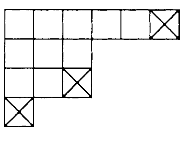

Fig. 2. Young diagram for $(6, 3^2, 1)$.

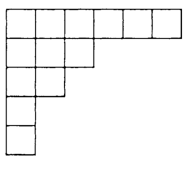

Fig. 3. Young diagram for $(6, 3, 2, 1^2)$.

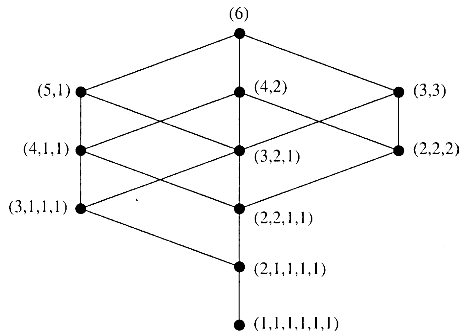

Fig. 4. Partial order ($\preceq$) on the partitions of 6.

all immanants. Of course, it would be sufficient to show the weaker hypothesis that combining any other part with the largest part increases the ranking of the partition in the partial order. The truth of the above conjectures is easily established for $n \leqslant 7$ from the lattice diagrams given by Pate [77]. As an example, Fig. 4 shows the known relationships between the partitions of 6. In this figure, a line links two partitions α and β to indicate that $\beta \preceq \alpha$ if β is lower down the page than α. This diagram is complete except that it has not yet been established which, if any, of the partitions $(4, 2)$, (3^2) or (2^3) is dominated by $(5, 1)$. It can be seen that in every case the combination of two parts increases the partition and that the only relationship in Fig. 4 not predicted by this observation is that $(2^3) \preceq (3^2)$.

To close this section we briefly mention some related developments.

(1) Questions of a similar nature to the permanental dominance conjecture (which applies to positive semi-definite Hermitian matrices) have been asked about the class of totally positive matrices (real matrices with non-negative minors). See [94,95] for details.

(2) Chan and Lam [9] sharpened the inequalities in (2) in the case of matrices which are the Laplacians of trees.

4. Extremes of the permanent on Λ_n^k

Another area in which extensive progress has recently been made is in our understanding of the matrices which achieve extremal values of the permanent in Λ_n^k. These matrices are of direct relevance to Conjectures 5, 6, 23, 24, 25, 26 and Problems 3, 4, 10, 11, 12.

Let us begin with the question of minimising the permanent on Λ_n^k. For $1 \leqslant k \leqslant n \leqslant 11$ the minimum values of the permanent in Λ_n^k are given in Table 1. These values were established by computer enumeration [104]. We use a prime ($'$) to mark values which are not achieved by any circulant matrix in the appropriate class. For

Table 1
Minimum value of per(A) for $A \in \Lambda_n^k$

k	n									
	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2
3	–	6	9	12′	17	24	33	42′	60′	83′
4	–	–	24	44	80	144	248′	440′	764′	1316′
5	–	–	–	120	265	578′	1249	2681	5713	12105
6	–	–	–	–	720	1854	4738	12000′	30240′	75510
7	–	–	–	–	–	5040	14833	43386′	126117′	364503
8	–	–	–	–	–	–	40320	133496	439792	1441788′
9	–	–	–	–	–	–	–	362880	1334961	4890740′
10	–	–	–	–	–	–	–	–	3628800	14684570
11	–	–	–	–	–	–	–	–	–	39916800

$n \leqslant 11$, this answers Problem 11 from [69] which asks whether the minimal value of the permanent in Λ_n^k is achieved by a circulant matrix.

An important theorem was obtained in 1998 by Schrijver [83], who showed that:

Theorem 4.1. *For any integers $n \geqslant k \geqslant 1$ and any $A \in \overline{\Lambda}_n^k$,*

$$\operatorname{per}(A) > \left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n. \tag{3}$$

For any given k, the base $(k-1)^{k-1}/k^{k-2}$ is best possible, in the sense that

$$\lim_{n\to\infty}\left(\min_{A\in\overline{\Lambda}_n^k}\operatorname{per}(A)\right)^{1/n} = \frac{(k-1)^{k-1}}{k^{k-2}}. \tag{4}$$

Wanless [103] then used this result to show:

Theorem 4.2. *Theorem 4.1 holds with Λ_n^k in place of $\overline{\Lambda}_n^k$.*

The above results give useful information about the minimum permanent in the sparse case, when $k \ll n$. There is also something known about the dense case, when $n - k \ll n$.

Henderson [34] found the minimum permanent among all (0, 1)-matrices with at most two zeroes in any row or column, without necessarily finding all matrices which achieve this permanent. McKay and Wanless [61] obtained a complete characterisation of the matrices which minimise the permanent in Λ_n^{n-2}. Let C_m denote the circulant (0, 1)-matrix of order m defined by $C_m = I_m + P_m$.

Theorem 4.3. *The minimum permanent in Λ_n^{n-2} is achieved by the complement of*

$$\begin{cases} C_n & n \leqslant 4, \\ C_t \oplus C_{t+1} & n = 2t+1 \geqslant 5, \\ C_6 \text{ or } C_4 \oplus C_2 \text{ or } C_2 \oplus C_2 \oplus C_2 & n = 6, \\ C_n \text{ or } C_{t-1} \oplus C_{t+1} & n = 2t \geqslant 8. \end{cases}$$

Any matrix in Λ_n^{n-2} not permutation equivalent to one of the above matrices has a strictly higher permanent.

A very important asymptotic result, due to Godsil and McKay [27], allows us to estimate the permanent of dense (0, 1)-matrices with an equal number of zeroes per row and column. Suppose that $0 \leqslant k = \mathrm{O}(n^{1-\delta})$ for a constant $\delta > 0$ as $n \to \infty$. Then

$$\operatorname{per}(A) = n!\left(\frac{n-k}{n}\right)^n \exp\left[\frac{k}{2n} + \frac{3k^2-k}{6n^2} + \frac{2k^3-k}{4n^3} + \frac{15k^4+70k^3-105k^2+32k}{60n^4} + \frac{z}{n^4} + \frac{2z(2k-1)}{n^5} + \mathrm{O}\left(\frac{k^5}{n^5}\right)\right] \tag{5}$$

for all $A \in \Lambda_n^{n-k}$, where z denotes the number of 2×2 submatrices of A which contain only zeroes. In particular, if $0 \leqslant k = \mathrm{O}(n^{1-\delta})$ for a constant $\delta > 0$ as $n \to \infty$ then (5) shows that $\operatorname{per}(A)$ is asymptotically equal to $n!(1-k/n)^n$ for all $A \in \Lambda_n^{n-k}$, but also that the permanent will be minimised by some matrix which minimises z.

Correspondingly, (5) can be applied to the problem of maximising the permanent in Λ_n^k, which we consider next. It shows that for sufficiently dense matrices the permanent will be maximised by some matrix which maximises z.

For $1 \leqslant k \leqslant n \leqslant 11$ the maximum values of the permanent in Λ_n^k are given in Table 2. These values were established by computer enumeration [61]. We use a prime ($'$) to mark values which are not achieved by any circulant matrix in the appropriate class. For $n \leqslant 11$, this helps to answer Problem 12 from [69] which asks whether the maximal value of the permanent in Λ_n^k is achieved by a circulant matrix.

It is known that for $n = tk + r$ with $0 \leqslant r < k$,

$$(k!)^t r! \leqslant \max_{A \in \Lambda_n^k} \operatorname{per}(A) \leqslant (k!)^{n/k}. \tag{6}$$

The upper bound is a classical result due to Brègman [6] and the lower bound is due to Wanless [101], who showed that it implies that

$$\left(\max_{A \in \Lambda_n^k} \operatorname{per}(A)\right)^{1/n} \sim (k!)^{1/k} \tag{7}$$

whenever $k = \mathrm{o}(n)$.

Table 2
Maximum value of per(A) for $A \in \Lambda_n^k$

k	n									
	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1
2	2	2	4	$4'$	8	$8'$	16	$16'$	32	$32'$
3	–	6	9	13	36	$54'$	81	216	$324'$	$486'$
4	–	–	24	44	82	$148'$	576	$1056'$	$1968'$	$3608'$
5	–	–	–	120	265	$580'$	1313	$2916'$	14400	$31800'$
6	–	–	–	–	720	1854	4752	$12108'$	32826	$86400'$
7	–	–	–	–	–	5040	14833	$43424'$	$127044'$	$373208'$
8	–	–	–	–	–	–	40320	133496	440192	$1448640'$
9	–	–	–	–	–	–	–	362880	1334961	$4893072'$
10	–	–	–	–	–	–	–	–	3628800	14684570
11	–	–	–	–	–	–	–	–	–	39916800

So much for the maximum value of the permanent. We now discuss what is known about the structure of those matrices which achieve this permanent. We define,

$$M_n^k = \left\{A \in \Lambda_n^k : \mathrm{per}(A) \geqslant \mathrm{per}(B) \text{ for all } B \in \Lambda_n^k\right\}.$$

We define a *component* of a matrix A to be a maximal fully indecomposable submatrix of A. Each $A \in \Lambda_n^k$ is permutation equivalent to the direct sum of its components. Moreover, the permanent of A is the product of the permanents of its components, from which we deduce that each component must be chosen to maximise its own permanent. Wanless [100] also showed that:

Theorem 4.4. *For each integer k there is an integer ν_k such that for every $n > \nu_k$ we have $A \in M_n^k$ if and only if $A \oplus J_k \in M_{n+k}^k$.*

In particular this shows that the permanent is maximised by taking small components (in the sense that their order is bounded by some function of k only), most of which are copies of J_k. Regarding this last point, the proof of Theorem 4.4 showed that fewer than k of the components can differ from J_k.

An interesting parallel can be found by considering matrices in the set

$$\overline{M}_n^k = \left\{A \in \Lambda_n^k : \mathrm{per}(\overline{A}) \geqslant \mathrm{per}(\overline{B}) \text{ for all } B \in \Lambda_n^k\right\}.$$

Hence the matrices in $\overline{M}_n^k$ are simply the complements of the matrices in M_n^{n-k}. Wanless [100] showed that, analogously to Theorem 4.4:

Theorem 4.5. *For each integer k there is an integer $\bar{\nu}_k$ such that for every $n > \bar{\nu}_k$ we have $A \in \overline{M}_n^k$ if and only if $A \oplus J_k \in \overline{M}_{n+k}^k$.*

Indeed the sets M_n^k and $\overline{M}_n^k$ sometimes coincide. McKay and Wanless [61] showed that:

Theorem 4.6. *If $n = mk$ for an integer $m \geqslant 5$, then $M_n^k = \overline{M}_n^k$.*

Note that in this case the exact composition of M_n^k is known from Brègman's theorem. It is slightly surprising that Theorem 4.6 is not true for $m = 3$. Indeed, Theorem 4.7 provides a counterexample when $n = 6$ and $k = 2$. Another counterexample is known [61] for $n = 9$, $k = 3$. The truth of Theorem 4.6 for $m = 4$ has not been resolved.

Results of a similar nature to Theorem 4.6 have been obtained for $k = 2$ and $k = 3$ and arbitrary n. Brualdi et al. [8] showed that (see also [61]):

Theorem 4.7. *For $5 \leqslant n \leqslant 7$, the set $\overline{M}_n^2$ consists of the matrices which are permutation equivalent to*

$$\begin{cases} C_5 & n = 5, \\ C_3 \oplus C_3 & n = 6, \\ C_5 \oplus C_2 \text{ or } C_3 \oplus C_2 \oplus C_2 & n = 7. \end{cases}$$

For any other n, $\overline{M}_n^2 = M_n^2$, meaning that it consists of the matrices with the maximum possible number of components.

Also, Wanless [100] showed that $M_n^3 = \overline{M}_n^3$ if $n \equiv 0, 1 \bmod 3$ and n is large, although this is not true when $n \in \{7, 9\}$. By contrast, for all large $n \equiv 2 \bmod 3$ the sets M_n^3 and $\overline{M}_n^3$ are disjoint.

Theorem 4.4 showed that components of maximising matrices cannot, informally speaking, be "too big". However, Wanless [101] has shown that J_k and matrices permutation equivalent to D_{k+1} are the only common small components in the following sense.

Theorem 4.8. *For each given integer $r \geqslant 2$ there exist finite sets K_r and N_r with the property that it is impossible to find $k \notin K_r$, $n \notin N_r$ and $A \in M_n^k$ such that A has a component in Λ_{k+r}^k.*

Moreover, if $t = k - a$ where $1 \leqslant a = o(\log k)$ then for sufficiently large k, $\mathrm{per}(\sum^t D_{k+1}) < \mathrm{per}(X \oplus \sum^{t-1} J_k)$ for all $X \in \Lambda_{k+t}^k$. This gives an upper bound on the number of components which are copies of D_{k+1}. This bound conflicts with Conjecture 26, although Theorem 4.8 indicates that Merriell [63] was probably on the right track when he made that conjecture.

A key step in the proof of Theorem 4.8 was the following result, which may be of independent interest.

Theorem 4.9. *Let* a, b *satisfy* $0 \leqslant a < b - 1$ *and let* k *be sufficiently large. Then* $\mathrm{per}(U \oplus V) < \mathrm{per}(X \oplus Y)$ *and* $\mathrm{per}(\overline{U \oplus V}) < \mathrm{per}(\overline{X \oplus Y})$ *for every choice of* $U \in \Lambda_{k+a}^k$, $V \in \Lambda_{k+b}^k$, $X \in \Lambda_{k+a+1}^k$ *and* $Y \in \Lambda_{k+b-1}^k$.

In summary then, a lot is known about the matrices which achieve the extremal values of the permanent in Λ_n^k, but the problem looks to be sufficiently complicated that a complete solution may never be found. Part of the complication, as is evident from Theorems 4.3, 4.6 and 4.7, is that small examples frequently do not fit the general pattern and also that the structure of the optimal matrix in these cases is sometimes not unique up to permutation equivalence.

We close the section by mentioning some significant related developments.

(1) Soules [91,92,93] has recently obtained a number of upper bounds for the permanent of non-negative matrices. Each of his bounds reduces to the Brègman bound when applied to (0, 1)-matrices.

(2) Liang and Bai [53] recently obtained an upper bound for the permanent of (0, 1)-matrices. Their bound is inferior to the Brègman bound when applied to matrices in Λ_n^k. However, it improves on the Brègman bound for some (0, 1)-matrices whose row sums vary greatly.

5. Current status of conjectures

Conjectures 1, 2, 7, 8, 10, 11, 13, 14 and 16 were solved prior to 1986. We shall not say anything further about them.

Conjecture 3 (Marcus and Minc [59]). *If* $A \in \Omega_n$, $n \geqslant 2$, *then*

$$\mathrm{per}(A) \geqslant \mathrm{per}\left(\frac{n\mathscr{J}_n - A}{n-1}\right). \tag{8}$$

If $n \geqslant 4$, *equality can hold in* (8) *if and only if* $A = \mathscr{J}_n$.

Hwang [39] showed that (8) holds whenever A is partly decomposable.

Malek [56] proposes a generalization of (8) to sums of subpermanents; namely he conjectures that for each k,

$$\sigma_k(A) \geqslant \sigma_k\left(\frac{n\mathscr{J}_n - A}{n-1}\right). \tag{9}$$

He shows that this stronger conjecture is true for any normal doubly stochastic matrix A whose eigenvalues lie in the sector $[-\pi/2n, \pi/2n]$ of the complex plane and also for all A in a sufficiently small neighborhood of $\mathscr{J}_n$.

In a later paper [57] the same author showed that Conjecture 12 implies Conjecture 3, but that is a moot point now as the former conjecture has since been shown to be false.

Conjecture 4 (Wang [97]). *If $A \in \Omega_n$, and $n \geqslant 2$, then*

$$\operatorname{per}(A) \geqslant \operatorname{per}\left(\frac{n\mathscr{J}_n + A}{n+1}\right). \tag{10}$$

If $n \geqslant 3$, equality can hold in (10) if and only if $A = \mathscr{J}_n$.

Chang [10] and Foregger [24] independently proved the $n = 4$ case. Hwang [39] showed that (10) holds when A is partly decomposable.

Kopotun [50] has made the broader conjecture that (10) holds with σ_k in place of per for $k = 2, 3, \ldots, n$.

Conjecture 5 (Ryser in [68]). *If Λ_v^k contains incidence matrices of (v, k, λ)-configurations, then the permanent takes its minimum in Λ_v^k at one of these incidence matrices.*

Wanless [104] showed by computer enumeration that this conjecture is true for $v < 13$. Otherwise, no progress.

Conjecture 6 (Minc in [58]). *For a fixed v,*

$$\min\left\{\operatorname{per}\left(\frac{1}{k}A\right) \,\middle|\, A \in \Lambda_v^k\right\} \tag{11}$$

is monotone decreasing in k.

This conjecture was proved by Wanless [104] for matrices which are sufficiently small, sparse or dense. Its truth for $n \leqslant 11$ can easily be checked from Table 1. Wanless also showed that it is true for $k < \mathrm{o}(n^{1/4})$, using (4) and for $k > n - \mathrm{o}(n^{6/7})$, by using (5). Similar (though not identical) statements were proved with min replaced by max in (11).

Conjecture 9 (Marcus in [58]). *Let A be an $mk \times mk$ positive semi-definite Hermitian matrix partitioned into $k \times k$ blocks A_{ij}, $i, j = 1, 2, \ldots, m$. Let G be the $m \times m$ matrix whose (i, j) entry is $\operatorname{per}(A_{ij})$ for $i, j = 1, 2, \ldots, m$. Then*

$$\operatorname{per}(A) \geqslant \operatorname{per}(G). \tag{12}$$

If the A_{ii} are positive definite, then equality holds in (12) if and only if

$$A = A_{11} \oplus A_{22} \oplus \cdots \oplus A_{mm}. \tag{13}$$

No progress.

Conjecture 12 (Holens [36], Djokovic [17]). *If $A \in \Omega_n$, $A \neq \mathscr{J}_n$ and k is an integer, $1 \leqslant k \leqslant n$, then*

$$\sigma_k(A) > \frac{(n-k+1)^2}{nk}\sigma_{k-1}(A). \tag{14}$$

FALSE. Taking $k = n$ in (14) implies that the ratio $\sigma_{n-1}(A)/\sigma_n(A)$ is bounded by a quadratic in n. However, Wanless [99] showed that this ratio cannot be bounded by any polynomial in n. The same paper shows that (i) for any given j there exists a matrix $A \in \Omega_n$ for some sufficiently large n for which (14) fails for all $k > n - j$; (ii) it is possible for (14) to hold when $k = n$ but fail to hold for $k = n - 1$; (iii) there is a counterexample to (14) of order 22. The smallest order of a counterexample has not been established.

See also Problem 8 and Conjecture 3.

Conjecture 15 (Foregger [22]). *If A is a nearly decomposable matrix in Ω_n, then*

$$\mathrm{per}(A) \geqslant 1/2^{n-1}. \tag{15}$$

Equality holds in (15) *if and only if $A = \frac{1}{2}(I_n + P_n)$, up to permutations of rows and columns.*

No progress.

Conjecture 17 (Foregger in [68]). *For any positive integer n, there exists an integer $k = k(n)$ such that*

$$\mathrm{per}(A^k) \leqslant \mathrm{per}(A) \tag{16}$$

for all A in Ω_n.

Chang [11] proved this conjecture for $n = 3$. He also showed for arbitrary n that if $A \in \Omega_n$ and $\mathrm{per}(A) \geqslant \frac{1}{2}$ then $\mathrm{per}(A^k) \leqslant \mathrm{per}(A)$ for all $k \geqslant 1$.

Conjecture 18 (Merris [64]). *If $A \in \Omega_n$, then*

$$n\,\mathrm{per}(A) \geqslant \min_i \sum_{j=1}^{n} \mathrm{per}(A(j|i)). \tag{17}$$

No progress.

Conjecture 19 (Wang [96]). *If two $n \times n$ Hadamard matrices have the same permanent, then either matrix can be obtained from the other by some of the following operations*: (1) *permutations of rows or columns*, (2) *multiplications of rows or columns by* -1, (3) *transposition of the matrix.*

FALSE: Wanless [102] showed that the smallest counterexamples are of order 20. For that order there are exactly 3 equivalence classes of Hadamard matrices under the given operations and yet every Hadamard matrix H of order 20 satisfies $|\text{per}(H)| = 219414528$.

Conjecture 20 (Gyires [33]). *Let $A \in \Omega_n$. Then*

$$\frac{4(\text{per}(A))^2}{\text{per}(AA^*) + \text{per}(A^*A) + 2\text{per}(A^2)} \geqslant \frac{n!}{n^n}. \tag{18}$$

Equality holds in (18) if and only if $A = \mathscr{J}_n$.

Chang [11] proved this conjecture for $n = 3$. He also proved it for any $A = [a_{ij}] \in \Omega_n$ for which

$$\min_{i,j} a_{ij} \geqslant \frac{n-2}{(n-1)^2}.$$

Conjecture 21 (Flor [21]). *If A is a non-negative $n \times n$ matrix and k is any integer, $1 \leqslant k \leqslant n$, then*

$$\sum (\text{per}(B) - \text{per}(C))(s(B) - s(C)) \geqslant 0, \tag{19}$$

where B and C range independently over all $k \times k$ submatrices of A, and $s(X)$ denotes the sum of all the entries of matrix X.

FALSE. In the special case when A is doubly stochastic, (19) reduces to Conjecture 12, which is now known to be false.

Conjecture 22 (Sasser in [69]). *Let A be an $n \times n$ non-negative matrix. Then*

$$\frac{1}{n^2} \sum_{i,j=1}^{n} \delta_r(A(i|j))\delta_s(A(i|j)) \geqslant \delta_r(A)\delta_s(A), \tag{20}$$

where $r, s = 1, 2, \ldots, n$, and $\delta_r(A)$ denotes the average of all the r-diagonal products of A,

$$\delta_r(A) = \frac{1}{r!\binom{n}{r}^2} \sum_{\alpha,\beta \in Q_{n-r,n}} \text{per}(A(\alpha|\beta)).$$

FALSE. Again, if we specialise to the case when A is doubly stochastic and $s = 1$ then (20) reduces to Conjecture 12.

Conjecture 23 (Schrijver and Valiant [84]). *Let*

$$\lambda_k(n) = \min \left\{ \text{per}(A) \mid A \in \Lambda_n^k \right\}.$$

Then

$$\lim_{n\to\infty} (\lambda_k(n))^{\frac{1}{n}} = \frac{(k-1)^{k-1}}{k^{k-2}}. \tag{21}$$

TRUE. When Schrijver and Valiant [84] originally posed their conjecture it referred to the minimum permanent in $\overline{\Lambda}_n^k$, not the minimum permanent in Λ_n^k. Schrijver [83] proved the original conjecture and Wanless [103] proved Minc's mistaken version of it (as given above), see Theorems 4.1 and 4.2.

Conjecture 24 (Minc [69]). *The permanent function attains its minimum in $\overline{\Lambda}_n^k$ at a $(0, 1)$-matrix, i.e.,*

$$\min\left\{\operatorname{per}(A) \mid A \in \overline{\Lambda}_n^k\right\} = \min\left\{\operatorname{per}(A) \mid A \in \Lambda_n^k\right\}. \tag{22}$$

No progress, although (3) gives a lower bound for the minimum value of the permanent in $\overline{\Lambda}_n^k$.

Conjecture 25 (Merriell [63]). *Suppose $k \leqslant n \leqslant 2k$. The maximum permanent in Λ_n^k is*

$$\operatorname{per}\begin{bmatrix} J_{r,r} & L_{r,k} \\ L_{r,k} & J_{r,r} \end{bmatrix} \tag{23}$$

if $n = 2r$ is even; and it is

$$\operatorname{per}\begin{bmatrix} J_{r+1,r} & L_{r+1,k+1} \\ L_{r,k-1} & J_{r,r+1} \end{bmatrix}, \tag{24}$$

if $n = 2r + 1$ is odd and $k \geqslant 5$, where $L_{c,d}$ is a matrix in Λ_c^{d-c} satisfying

$$\operatorname{per}(L_{c,d}) = \max\left\{\operatorname{per}(A) \mid A \in \Lambda_c^{d-c}\right\}.$$

FALSE. A counterexample was given by Bol'shakov [5] in the case $n = 9, k = 7$. However, the comments at the end of Section 4 show that one small counterexample is not enough reason to completely abandon a conjecture of this nature. Merriell's conjecture is easily repaired to avoid this counterexample by simply raising the lower bound on k.

Nevertheless, Conjecture 25 is fatally flawed. Theorem 4.7 shows that both (23) and (24) hold when $k = n - 2$ for large $n \not\equiv 2 \bmod 4$. However, (23) fails when $k = n - 2$ for all large $n \equiv 2 \bmod 4$. Similarly, Theorem 4.6 shows that (23) and (24) are both false whenever $n = m(n - k)$ for an odd integer $m > 10$, although (23) is true for $n = m(n - k)$ for an even integer $m \geqslant 10$.

Conjecture 26 (Merriell [63]). *Let $n = kq + r$, where $0 \leqslant r \leqslant q$ if $1 \leqslant q < k - 3$, and $0 \leqslant r \leqslant k - 3$ if $q \geqslant k - 3$. Then the maximum permanent in Λ_n^k is*

$$\operatorname{per}\left(\sum^{q-r} J_k \oplus \sum^{r} D_{k+1}\right) = (k!)^{q-r}\left(n!\sum_{t=0}^{k+1}\frac{(-1)^i}{i!}\right)^r. \tag{25}$$

If $r = k - 2$ *then the maximum permanent in* Λ_n^k *is*

$$\operatorname{per}\left(\sum^{q-1} J_k \oplus \begin{bmatrix} J_{k-1} & I_{k-1} \\ I_{k-1} & J_{k-1} \end{bmatrix}\right) = (k!)^{q-1} \sum_{i=0}^{q-1} \left(\binom{k-1}{i} (k-1-i)! \right)^2 \tag{26}$$

and if $r = k - 1$ *then the maximum permanent in* Λ_n^k *is*

$$\operatorname{per}\left(\sum^{q-1} J_k \oplus \begin{bmatrix} J_{k-1,k} & O \\ I_k & J_{k,k-1} \end{bmatrix}\right) = (k!)^q (k-1)!. \tag{27}$$

FALSE. A counterexample to (25) in the case $n = 14$, $k = 5$ was given by Zagaglia-Salvi [105]. It follows that (25) fails for $n = 9 + 5t$, $k = t$ for all $t \geqslant 1$.

Also, as we saw in Section 4 there can never be more than $k - o(\log k)$ copies of D_{k+1}, so (25) is incorrect when r is close to k and k is large. However there is some evidence to suggest that it is correct when r is small compared to k.

It is also worth remarking that (27) fails when $n = 9$, $k = 5$ because it predicts that the maximum permanent is 2880, whereas it is actually 2916. The unique (up to permutation equivalence) matrix achieving this value is

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

When Conjecture 26 was originally posed in [63] it included a restriction that $k \geqslant 5$, which was omitted in [69]. Without this restriction there is another counterexample to (27) when $n = 7$, $k = 4$. There are no known counterexamples to (26). Note that for $q = 1$ and large k, Theorem 4.8 shows that (26) gives the largest permanent of any matrix whose complement is decomposable.

Conjecture 27 (Nemeth et al. [72]). *The permanents of* $p \times p$ $(0, 1)$*-circulants, for a prime* p, *attain* $O(p)$ *distinct values.*

Bernasconi et al. [4] proved that for circulants with 3 positive entries in each row the permanent can take at most $\lceil p/6 \rceil$ different values. Hence the conjecture is true if the restriction is added that the number of positive entries per row cannot exceed 3.

However, it is unlikely that the general conjecture is true. Resta and Sburlati [82] make the following rival conjecture.

Conjecture 27′. *For any fixed $k \geqslant 3$ the permanent takes $p^{k-2}/k! + \mathrm{O}(p^{k-3})$ different values on the circulants in Λ_p^k when p is prime.*

Conjecture 28 (Dittert in [69]). *Let K_n be the set of non-negative $n \times n$ matrices with the sum of their entries equal to n. Define function ϕ on $n \times n$ matrices by*

$$\phi(A) = \prod_{i=1}^{n} r_i + \prod_{j=1}^{n} c_j - \mathrm{per}(A) \tag{28}$$

where $r_1, r_2, \ldots, r_n$ and $c_1, c_2, \ldots, c_n$ are the row and column sums of A, respectively. Then

$$\max\{\phi(A) \mid A \in K_n\} = 2 - \frac{n!}{n^n}, \tag{29}$$

and the maximum is attained only for $A = \mathscr{J}_n$.

Hwang [38] proved the $n = 3$ case. He also showed that if $A = [a_{ij}]$ is a ϕ-maximising matrix in K_n then for $1 \leqslant i, j \leqslant n$,

$$\prod_{k \neq i} r_k + \prod_{k \neq j} c_k - \mathrm{per} A(i|j) \leqslant \phi(A)$$

with equality holding if $a_{ij} > 0$.

Cheon and Hwang [12] proposed a conjecture generalizing both the Tverberg-Friedland theorem and Conjecture 28: For any $A \in K_n$ and any $k \in \{1, \ldots, n\}$,

$$\sum_{\alpha,\beta \in Q_{k,n}} \left(\prod_{i \in \alpha} r_i + \prod_{j \in \beta} c_j - \mathrm{per}(A[\alpha|\beta]) \right) \leqslant \binom{n}{k}^2 \left(2 - \frac{k!}{n^k} \right) \tag{30}$$

with equality holding if and only if $k = 1$ or $A = \mathscr{J}_n$. They proved this conjecture for $n \leqslant 3$ and for $k \leqslant 2$ with no restriction on n.

Conjecture 29 (Wang [98]). *If $B \in \Omega_n$, $n \geqslant 3$, and*

$$\mathrm{per}(\theta B + (1-\theta)A) \leqslant \theta \mathrm{per}(B) + (1-\theta)\mathrm{per}(A) \tag{31}$$

for all $A \in \Omega_n$ and all $\theta \in [0, 1]$, then B is a permutation matrix.

FALSE. Karuppan Chetty and Maria Arulraj [47] gave a counterexample for $n = 3$, and proposed the following modified conjecture, which they proved for $n = 3$:

Conjecture 29′. (31) *holds for all $A \in \Omega_n$ and all $\theta \in [0, 1]$ if and only if B is permutation equivalent to the direct sum of an identity matrix and some number of 2×2 doubly stochastic matrices.*

Earlier, Kopotun [50] considered whether (31) holds with σ_k in place of per. He obtained some partial results in the case when $k \leqslant 3$ or $B = \mathscr{J}_n$.

Conjecture 30 (Chollet [13]). *If A and B are positive semi-definite Hermitian $n \times n$ matrices, then*

$$\operatorname{per}(A * B) \leqslant \operatorname{per}(A)\operatorname{per}(B). \tag{32}$$

Marcus and Sandy [60] noted that the $n = 3$ case of this conjecture (which had already been proved by Gregorac and Hentzel [29]) follows immediately from the proof, by Bapat and Sunder [2], of conjecture 31 for $n = 3$.

See also Conjecture 38.

Conjecture 31 (Soules in [69]). *Let $A = [a_{ij}]$ be a positive semi-definite Hermitian $n \times n$ matrix. Let B be the $n!$-square matrix whose (σ, τ) entry is $\prod_{t=1}^{n} a_{\sigma(t),\tau(t)}$, where σ and τ run over all permutations in S_n. Then* per(A) *is the maximal eigenvalue of B.*

Pate [73] proved a special case of this conjecture in order to show that Conjecture 42 holds for immanants associated with two part partitions, see Section 3.

Soules [90] showed that if Conjecture 31 is false for real matrices then for the smallest order for which it fails there must be a counterexample which is singular, has zero row sums and has several other properties.

Conjecture 32 (Bapat and Sunder [2]). *Let c be a complex valued function on S_n satisfying*

$$\sum_{\sigma,\tau \in S_n} x(\tau)^* c(\sigma\tau^{-1}) x(\sigma) \geqslant 0 \tag{33}$$

for all complex valued functions x on S_n. Then

$$c(\mathbf{1}_n)\operatorname{per}(A) \geqslant \sum_{\sigma \in S_n} c(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \tag{34}$$

for any positive semi-definite Hermitian $n \times n$ matrix $A = [a_{ij}]$.

This conjecture is equivalent to Conjecture 31 (see Fig. 1).

Conjecture 33 (Mehta [62]). *Let D be a fixed non-negative diagonal matrix. Then the maximum of* per$(U^* DU)$, *when U runs over all unitary matrices, is attained when all the main diagonal entries of $U^* DU$ are equal.*

FALSE. Drew and Johnson [19] give the following counterexample:

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad U^* DU = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 5 & 0 \\ 2 & 0 & 3 \end{pmatrix}.$$

The given matrix U^*DU achieves a permanent of 65, whereas every matrix of the form U^*DU with constant main diagonal has a permanent of $64\frac{1}{9}$.

Conjecture 34 (Lih and Wang [55]). *If $A \in \Omega_n$ and $\alpha \in [\frac{1}{2}, 1]$, then*

$$\operatorname{per}(\alpha \mathscr{J}_n + (1-\alpha)A) \leqslant \alpha \operatorname{per}(\mathscr{J}_n) + (1-\alpha)\operatorname{per}(A). \tag{35}$$

Foregger [24] proved the case $n = 4$.

Conjecture 35 (Kim and Roush [48]). *The maximum value of* $\operatorname{per}(I - A)$ *for* $A \in \Omega_{2k+1}$ *is* $3 \cdot 2^{k-2}$. *This value is attained for the direct sum of*

$$\frac{1}{2}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \tag{36}$$

and $k - 1$ *copies of* $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

No progress.

Conjecture 36 (Kräuter [51]). *The minimum value of the permanent function on the set of n-square* $(1, -1)$*-matrices with positive permanent is* $2^{n-\lfloor \log_2(n+1) \rfloor}$.

Kräuter and Seifter [52] showed that the permanent cannot have a lower positive value than $2^{n-\lfloor \log_2(n+1) \rfloor}$. Hence to prove the conjecture it suffices to offer a construction which achieves this value. Wanless [102] has done this for $n \leqslant 20$.

Conjecture 37 (Kräuter [51]). *Let A be an n-square* $(1, -1)$*-matrix*, $n \geqslant 5$, *of rank* $r + 1$. *Then*

$$|\operatorname{per}(A)| \leqslant \operatorname{per}(C(n, r)), \tag{37}$$

where $C(n, r)$ is the n-square $(1, -1)$*-matrix whose first r main diagonal entries are* -1, *and all its other entries are equal to* 1. *Equality holds in* (37) *if and only if A can be obtained from $C(n, r)$ by a sequence of the following operations*: *interchange of two rows or columns*, *transposition*, *multiplication of a row or column by* -1.

No progress.

Conjecture 38 (Bapat and Sunder [1]). *If A and $B = [b_{ij}]$ are positive semi-definite Hermitian $n \times n$ matrices*, *then*

$$\operatorname{per}(A * B) \leqslant \operatorname{per}(A) \prod_{i=1}^{n} b_{ii}. \tag{38}$$

A correlation matrix is a positive semi-definite Hermitian matrix in which every entry on the diagonal is one. Zhang [106] showed that Conjecture 38 is true if and only if it is true for all correlation matrices A and B. Zhang also showed that it is true if A and B are correlation matrices and every off-diagonal entry of B is equal to some fixed t in the interval [0, 1].

Beasley [3] makes the following conjecture, which is stronger than Conjecture 30 and weaker than Conjecture 38: if $A = [a_{ij}]$ and $B = [b_{ij}]$ are positive semi-definite Hermitian $n \times n$ matrices then

$$\operatorname{per}(A * B) \leqslant \max\left\{\operatorname{per}(A)\prod_{i=1}^{n} b_{ii}, \operatorname{per}(B)\prod_{i=1}^{n} a_{ii}\right\}. \tag{39}$$

He proved this conjecture holds for $n = 2, 3$ as well as showing that (39) is true if and only if it holds for all correlation matrices.

Conjecture 39 (Minc [70]). *If $A = [a_{ij}]$ and $B = [b_{ij}]$ are positive semi-definite Hermitian $n \times n$ matrices, then*

$$\operatorname{per}(A * B) + \operatorname{per}(A)\operatorname{per}(B) \geqslant \operatorname{per}(A)\prod_{i=1}^{n} b_{ii} + \operatorname{per}(B)\prod_{i=1}^{n} a_{ii}. \tag{40}$$

TRUE. Jiao [44] proved this conjecture and showed that equality holds in (40) if and only if A or B is either a diagonal matrix or a matrix with a zero row/column.

Conjecture 40 (Bapat and Sunder [2]). *If A is positive definite, then* $\operatorname{per}(A)$ *is the largest eigenvalue of the matrix* $[a_{ij}\operatorname{per}(A(i|j))]$.

No progress.

Conjecture 41 (Foregger and Sinkhorn [25]). *If A is a nearly decomposable matrix minimizing the permanent in $\Omega(Z)$, and $(i, j) \in Z$, then* $\operatorname{per}(A(i|j)) > \operatorname{per}(A)$ *implies that (i, j) is a tie point for A.*

Foregger [23] proved a special case of this conjecture where the matrix A is associated with a type of bipartite graph which he called a *complex*. A complex in this sense consists of two special vertices which are joined by a number of separate paths.

Conjecture 42 (Lieb [54]). *Let G be a subgroup of S_n, and let χ be a character of G. Then*

$$\operatorname{per}(A)\chi(\mathbf{1}_n) \geqslant \sum_{\sigma\in G}\chi(\sigma)\prod_{i=1}^{n} a_{i\sigma(i)} \tag{41}$$

for any positive semi-definite Hermitian $n \times n$ matrix $A = [a_{ij}]$.

There has been substantial progress on this problem, particularly in the case when $G = S_n$. For details, see Section 3.

Conjecture 43 (Merris [66]). *Let $A = [a_{ij}]$ be a positive semi-definite Hermitian $n \times n$ matrix, $n \geqslant 4$. Then*

$$(n-1)\mathrm{per}(A) + \det(A) \geqslant n \prod_{i=1}^{n} a_{ii}. \tag{42}$$

TRUE. This was confirmed by Grone and Pierce [32]. They also characterised the cases when equality holds in (42), which for $n \geqslant 4$ only happens if A is diagonal or A has a zero row.

Also relevant is the paper of Grone and Merris [31] which included Conjecture 43 and a number of other unsolved problems in the area. Several of these were resolved by Grone and Pierce [32], but others remain open.

Conjecture 44 (Folklore, see [70]). *The permanent function on the set of $n \times n$ doubly stochastic matrices with zero trace achieves its minimum uniquely at the matrix all of whose off-diagonal entries are* $1/(n-1)$.

No progress.

6. Current status of open problems

Problem 1 (*Marcus and Minc* [58]). Find the maximum value of $\mathrm{per}(U^*AU)$ if A is a fixed n-square positive semi-definite Hermitian matrix, $n \geqslant 3$, and U runs over all $n \times n$ unitary matrices.

Drew and Johnson [18] solved this problem for $n = 3$. They showed that the maximum permanent is always achieved by a persymmetric matrix (one which is symmetric in both the main left-to-right diagonal and the main right-to-left diagonal), and posed the question as to whether this is true more generally.

Grone et al. [30] obtained a necessary condition for $B = U^*AU$ to maximise its permanent (for a given A); namely B must commute with the permanental adjoint of B. (If $B = [b_{ij}]$ the permanental adjoint of B has $\mathrm{per}(B(j|i))$ as its entry in row i, column j.)

Problem 2 (*Marcus and Minc* [58]). Let H be a subgroup of S_n, and let χ be a character of degree 1 of H. Under what conditions on χ does the inequality

$$\sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} \leqslant \mathrm{per}(A) \tag{43}$$

hold for all positive semi-definite Hermitian A.

This problem is very closely related to Conjecture 42, see Section 3.

Problem 3 (*Greenstein in* [68]). Find all the values that per(A) can take for A in Λ_n^3.

As mentioned in discussing Conjecture 27, Bernasconi et al. [4] and Resta and Sburlati [82] have investigated the possible values that the permanent can take on the circulants in Λ_n^3. Also, Codenotti and Resta [14] have shown that the permanent of circulants in Λ_n^3 can be calculated quickly by writing it as a linear combination of four determinants. In contrast, Dagum and Luby [15] have shown that the problem of calculating the permanent of a general matrix in Λ_n^3 is $\#P$-complete, meaning it is extremely unlikely that a polynomial-time algorithm can be found. This last result dampens hopes of a simple answer to Problem 3 being found in the near future.

Problem 4 (*Minc* [68]). Find the maximal value of per(A) in Λ_n^k in case k does not divide n.

There has been substantial progress on this problem. For details see Section 4.

Problem 5 (*Wang* [96]). Can the permanent of an $n \times n$ Hadamard matrix vanish for $n > 2$?

Wanless [102] showed that the answer is negative for $2 < n < 32$.

Problem 6 (*Wang* [96]). Find a significant upper bound for per(A) in the set of non-singular n-square (1,-1)-matrices.

When this problem was originally stated in [68] the word "non-singular" was accidentally omitted, but this mistake was corrected in [70]. Also note that Conjecture 37 suggests what the optimising matrices might be.

Problem 7 (*Wang* [96]). For what values of n do there exist non-singular n-square (1,-1)-matrices A such that $|\text{per}(A)| = |\det(A)|$?

This problem has been slightly reworded. In its original form it was rather trivially answered by the non-existence of a solution for $n = 4$. It was noted in [70] that there exists no solution when $n \in \{2, 3, 4\}$ or $n = 2^k - 1$ for an integer $k \geqslant 2$, but that there is a solution when $n \in \{5, 6\}$. Wanless [102] showed that there are solutions for all $n \in \{8, 9, \ldots, 20\} \setminus \{15\}$.

Problem 8 (*Friedland and Minc* [26]). Find matrices A on the boundary of Ω_n so that the permanent is monotone increasing on the segment $(1-\theta)\mathscr{J}_n + \theta A$, $0 \leqslant \theta \leqslant 1$.

Foregger [24] showed that several classes of matrices in Ω_4 have the required monotonicity property. Hwang [40] proved monotonicity for all matrices of the form $\mathscr{J}_{n_1} \oplus \cdots \oplus \mathscr{J}_{n_t}$, thereby answering a challenge posed by Lih and Wang. He also showed monotonicity for $(D_m \otimes s\mathscr{J}_s)/(ms - s)$, which generalises an earlier result of Friedland and Minc.

The counterexamples to Conjecture 12 all provide examples of matrices which do *not* have the required monotonicity property.

Goldwasser [28] showed the non-trivial result that if two doubly stochastic matrices satisfy (14) for all k then so does their direct sum. Kopotun [49] showed that (14) holds for $k = 4$ provided $n \geqslant 5$.

Problem 9 (*Minc* [68]). Find a positive number $b = b(n)$ such that $\mathrm{per}(A) \geqslant n!/n^n$ for any $A \in \Omega_n$ satisfying $\|A - \mathscr{J}_n\| \leqslant b$.

This problem was completely solved prior to the period of the current survey (see [69]). However, see Problem 13.

When the next problem was quoted in [69] there was a typographical error. The 2.99 in the upper bound mistakenly appeared as 2.29.

Problem 10 (*Minc* [68]). Find numbers m and M such that

$$2.31^n < m^n \leqslant \mathrm{per}(A) \leqslant M^n < 2.99^n \tag{44}$$

for all $A \in \Lambda_n^6$ and sufficiently large n. Alternatively, find m and M that satisfy (44) for all circulants in Λ_n^6 for sufficiently large n.

The first part of this problem is completely solved. By Theorem 4.2, the largest possible value for m is $5^5/6^4 \approx 2.41$. Also by (7),

$$\lim_{n\to\infty} \left(\max_{A\in\Lambda_n^6} \mathrm{per}(A) \right)^{1/n} = 6!^{1/6} \approx 2.99 \tag{45}$$

which means that there is no constant M which satisfies (44). In summary then,

$$(5^5/6^4)^n \leqslant \mathrm{per}(A) \leqslant (6!^{1/6})^n \tag{46}$$

for all $A \in \Lambda_n^6$ and any n, however large. The constants in both the upper and lower bound are best possible. Nor can the upper bound in (46) be improved by restricting attention to circulants, since whenever $n \equiv 0 \bmod 6$, the upper bound is actually achieved by a circulant (see Problem 12). Thus the only issue remaining is whether the lower bound in (46) can be improved when A is restricted to being a circulant.

Problem 11 (*Minc* [69]). Does there exist a matrix in Λ_n^k whose permanent is strictly smaller than that of any circulant in Λ_n^k?

The answer for $n \leqslant 11$ is given in Table 1.

Problem 12 (*Minc* [69]). If k does not divide n, find an upper bound L for the permanents of matrices in Λ_n^k, $L < (k!)^{n/k}$. Does there exist a matrix in Λ_n^k whose permanent is strictly greater than that of any circulant in Λ_n^k?

The answer to this last question is now known to generally, but not always, be "yes". If k divides n then Brègman's theorem (see (6)) implies that the maximum permanent is achieved by the circulant whose positive diagonals are equally spaced. Likewise, if $n = m(n-k)$ for an integer $m \geqslant 5$, then Theorem 4.6 shows that the maximum permanent is achieved by the circulant whose zero diagonals are equally spaced. However, the components in a circulant are all permutation equivalent to each other. Consequently, for the cases $k \ll n$ and $n-k \ll n$ respectively, Theorem 4.4 and Theorem 4.5 imply that the above examples are the only ones for which the maximum is achieved by a circulant. It may be true in the majority of these cases that each *component* of the maximising matrix is a circulant.

For $n \leqslant 11$, Table 1 shows when the maximum permanent in Λ_n^k is achieved by a circulant.

Problem 13 (*Sinkhorn* [86]). Determine the largest number $b = b(n)$ such that $\operatorname{per}(A) \geqslant n!/n^n$ for all real $n \times n$ matrices A all of whose row and column sums are equal to 1 and which satisfy $\|\mathscr{J}_n - A\| \leqslant b$.

No progress.

Problem 14 (*Brualdi* [7]). Characterize cohesive matrices.

See Problem 15.

Problem 15 (*Brualdi* [7]). Characterize barycentric matrices.

If a (0, 1)-matrix is barycentric then it is cohesive, but a (0, 1)-matrix can be cohesive without being barycentric. Brualdi [7] conjectured that C_n is a strong candidate for such a matrix, where C_n is the (0, 1)-matrix with 0's on the main diagonal except the (1, 1)-position and 1's elsewhere. Indeed, Song [88] proved that C_n is never barycentric for $n \geqslant 4$ and Hong et al. [37] showed that C_4 is cohesive. It is still open for $n \geqslant 5$ whether C_n is cohesive or not. Song [87] (also see [89]) gave some examples of cohesive matrices (some of which are barycentric, and some of which are not) by studying the minimum permanents on faces of Ω_n defined by fully indecomposable (0, 1)-matrices containing an identity matrix as a submatrix. Do and Hwang [16] obtained an example of a non-barycentric matrix by showing that if a (0, 1)-matrix D is barycentric then the minimum permanent over the face $\Omega(D)$ is a rational number.

Problem 16 (*Brualdi* [7]). Let A be an $n \times n$ (0, 1)-matrix. Is the set of all matrices in $\Omega(A)$ with minimum permanent a convex polyhedron? If not, is it connected?

SOLVED! Fischer and Hwang [20] investigated $\min(H_n)$, the set of all minimizing matrices over the face of Ω_n determined by $H_n := (I_n + P_n) \otimes K_2$, where P_n is the permutation matrix of order n with 1's in the position $(1, 2), (2, 3), \ldots,$

$(n-1, n)$ and $(n, 1)$, and K_2 is the all 1's matrix of order 2. As a result, they showed that $\min(H_n)$ is not connected and hence not convex, which solves this problem negatively.

Problem 17 (*Minc* [71]). Find an algorithm for the characteristic polynomial of the kth permanental compound of a given square matrix.

No progress.

Problem 18 (*Minc* [71]). Find an algorithm for the Perron root of the kth permanental compound of a given non-negative square matrix.

No progress.

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